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On Perfect Involution Groups

by

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1. Introduction

By Herglotz' Theorem, a sequence $(s_n)_{n\in\mathbf{Z}}$ of complex numbers is a trigonometric moment sequence if and only if (s_n) is positive definite in the sense that $\sum_{j,k=-n}^{n} c_j\overline{c_k}s_{j-k} \geq 0$ for every choice of $n \in \mathbf{N}_0$ and $c_{-n}, \cdots, c_n \in \mathbf{C}$. By a theorem of Jones, Njåstad, and Thron [10], a sequence $(s_n)_{n\in\mathbf{Z}}$ of reals is a two-sided moment sequence, in the sense that $s_n = \int_{\mathbf{R}\setminus\{0\}} x^n d\mu(x)$, $n \in \mathbf{Z}$, for some measure μ on $\mathbf{R}\setminus\{0\}$, if and only if (s_n) is positive definite in the semigroup sense, that is, $\sum_{j,k=-n}^{n} c_j c_k s_{j+k} \geq 0$ for every choice of $n \in \mathbf{N}_0$ and $c_{-n}, \cdots, c_n \in \mathbf{R}$.

The difference between the theorems of Herglotz and of Jones et al. is twofold: Firstly, there is the difference in the definition of positive definiteness. Secondly, the measure in Herglotz' Theorem is always uniquely determined, whereas in the theorem of Jones et al. there may be several measure with the moment sequence (s_n). Such is the case, for example, if $s_n = e^{n^2/2}$ ([3], 6.4.6).

To place the moment problems of Herglotz and of Jones et al. in a common frame, consider an arbitrary abelian semigroup with zero and involution $(S, +, *)$. A function $\varphi : S \to \mathbf{C}$ is *positive definite* if $\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_j + s_k^*) \geq 0$ for every choice of $n \in \mathbf{N}$ and $c_1, \cdots, c_n \in \mathbf{C}$. The set $\mathcal{P}(S)$ of all positive definite functions on S is a convex cone, stable under pointwise multiplication and closed under pointwise convergence in $\mathbf{C}^S$.

A *character* on S is a function $\sigma : S \to \mathbf{C}$ satisfying $\sigma(0) = 1$, $\sigma(s^*) = \overline{\sigma(s)}$, and $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. With pointwise multiplication, pointwise complex conjugation, and the topology of pointwise convergence, the set S^* of all characters on S is a topological involution semigroup with unit 1, the constant character.

Denote by $\mathcal{A}(S^*)$ the smallest σ-field of subsets of S^* rendering the evaluation $\sigma \mapsto \sigma(s) : S^* \to \mathbf{C}$ measurable for each $s \in S$, and by $F_+(S^*)$ the set of all measures defined on $\mathcal{A}(S^*)$ and integrating the evaluations. For $\mu \in F_+(S^*)$, define $\mathcal{L}\mu : S \to \mathbf{C}$ by $\mathcal{L}\mu(s) = \int_{S^*} \sigma(s)d\mu(\sigma)$, $s \in S$. A function $\varphi : S \to \mathbf{C}$ is a *moment function* if $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*)$, and a moment function φ is *determinate* if there is only one such μ. Denote by $\mathcal{H}(S)$ the set of all moment functions on S, and by $\mathcal{H}_D(S)$ the subset of determinate moment functions. We have $\mathcal{H}_D(S) \subset \mathcal{H}(S) \subset \mathcal{P}(S)$ since if $\mu \in F_+(S^*)$,

$s_1, \cdots, s_n \in S$, and $c_1, \cdots, c_n \in \boldsymbol{C}$ then

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \mathcal{L}\mu(s_j + s_k^*) = \int_{S^*} \left| \sum_{j=1}^{n} c_j \sigma(s_j) \right|^2 d\mu(\sigma) \geq 0 .$$

The $*$-semigroup S is *semiperfect* if $\mathcal{H}(S) = \mathcal{P}(S)$, and *perfect* if $\mathcal{H}_D(S) = \mathcal{P}(S)$.

Every abelian group with the involution $s^* = -s$ is perfect by the discrete version of the Bochner-Weil Theorem. More generally, a $*$-semigroup S is perfect if it is *$*$-divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $m, n \in \boldsymbol{N}_0$ such that $m + n \geq 2$ and $s = mt + nt^*$ ([8], Theorem 4). In particular, every rational vector space, with any involution whatsoever, is perfect. The perfectness of $\boldsymbol{Q}_+$ with its unique involution, the identity, was shown in [3], 6.5.6.

The semigroup $\boldsymbol{N}_0$ with its unique involution, the identity, is semiperfect by Hamburger's Theorem (see [1] or [3], 6.2.2), but is not perfect since there exist indeterminate moment sequences, such as $n \mapsto e^{n^2/2}$. As already mentioned, the group $\boldsymbol{Z}$ with the identical involution is semiperfect but not perfect.

For $k \geq 2$ the semigroups $\boldsymbol{N}_0^k$ and $\boldsymbol{Z}^k$ with the identical involution are not semiperfect. For $\boldsymbol{N}_0^k$ this was first shown in [2] and independently in [12]; see [3], 6.3.4. For $\boldsymbol{Z}^k$, see [3], 6.4.8. Also, the semigroup $\boldsymbol{N}_0^2$ with the involution $(m, n)^* = (n, m)$ is not semiperfect ([3], 6.3.5). Surprisingly, the group $\boldsymbol{Z}^2$ with the involution $(m, n)^* = (n, m)$ is semiperfect ([4], Theorem 1).

The purpose of the present paper is to contribute to the characterization of perfect groups with involution.

Consider S^* with the topology of pointwise convergence and denote by $\mathcal{B}(S^*)$ the Borel σ-field. Denote by $\mathcal{K}(S^*)$ the set of all compact subsets of S^*. A measure μ defined on $\mathcal{B}(S^*)$ is a *Radon measure* if $\mu(C) < \infty$ for all $C \in \mathcal{K}(S^*)$ and $\mu(B) = \sup\{\mu(C) | \mathcal{K}(S^*) \ni C \subset B\}$ for each $B \in \mathcal{B}(S^*)$. Denote by $E_+(S^*)$ the set of all Radon measures on S^* integrating the evaluations. Measures in $E_+(S^*)$ are bounded since they integrate the evaluation $\sigma \mapsto \sigma(0) = 1$. Bounded Radon measures being outer regular ([3], p. 17), measures in $E_+(S^*)$ are determined by their restriction to the topology. Since every Radon measure is τ-smooth ([3], 2.1.5), measures in $E_+(S^*)$ are even determined by their restriction to a base of the topology. Since $\mathcal{A}(S^*)$ contains such a base, the mapping $\mu \mapsto \mu | \mathcal{A}(S^*) : E_+(S^*) \to F_+(S^*)$ is one-to-one. We suppress this mapping and consider $E_+(S^*)$ as a subset of $F_+(S^*)$. A function $\varphi : S \to \boldsymbol{C}$ is a *Radon moment function* if $\varphi = \mathcal{L}\mu$ for some $\mu \in E_+(S^*)$, and a Radon moment function φ is *Radon determinate* if there is only one such μ. Denote by $\mathcal{H}_R(S)$ the set of all Radon moment functions on S, and by $\mathcal{H}_{R,D}(S)$ the subset of Radon determinate Radon moment functions. The $*$-semigroup S is *Radon semiperfect* if $\mathcal{H}_R(S) = \mathcal{P}(S)$, and *Radon perfect* if $\mathcal{H}_{R,D}(S) = \mathcal{P}(S)$.

Radon moment functions were called "moment functions" in [3]. Moment functions were introduced in [8] under the name of "quasi-moment functions".

If S is countable then S^* is a Polish space, $\mathcal{A}(S^*) = \mathcal{B}(S^*)$, and every bounded measure on $\mathcal{B}(S^*)$ is a Radon measure, so the concepts "Radon moment function" and so on are equivalent with their counterparts without the qualification "Radon". Every Radon perfect semigroup is perfect ([6], Corollary 4.1).

If S and T are $*$-semigroups, a mapping $h : S \to T$ is a *$*$-homomorphism* provided that $h(0) = 0$, $h(s^*) = h(s)^*$, and $h(s+t) = h(s) + h(t)$ for all $s, t \in S$. Every $*$-homomorphic image of a Radon perfect (Radon semiperfect) semigroup is Radon perfect (Radon semiperfect). See [3], 6.5.5, for the case of Radon perfectness, and [8], proof of Proposition 1, for the case of Radon semiperfectness. Every countable direct sum of Radon perfect semigroups is Radon perfect ([3], p. 224). Since $\boldsymbol{Q}$ is Radon perfect, it follows that every countable rational vector space carrying the identical involution is Radon perfect. An uncountable rational vector space carrying the identical involution, however, though perfect, is not Radon perfect, as can be seen from [6], Example 2.1.

Denote by W a rational vector space, the cardinality of which is the smallest uncountable cardinal number, carrying the identical involution. The main result of [5] states that an abelian group G with involution is Radon perfect if and only if neither W nor $\boldsymbol{Z}$ (with the identical involution) is a $*$-homomorphic image of G, and is Radon semiperfect if and only if neither W nor $\boldsymbol{Z}^2$ (with the identical involution) is a $*$-homomorphic image of G.

In order that an abelian group G be perfect, the condition that W not be a $*$-homomorphic image of G is *not* necessary since arbitrary rational vector spaces are perfect. The condition that $\boldsymbol{Z}$ (with the identical involution) not be a $*$-homomorphic image of G *is* necessary since $\boldsymbol{Z}$ is not perfect and since every $*$-homomorphic image a perfect semigroup is perfect ([8], Theorem 1). Whether the latter condition is also sufficient, we do not know.

A reduction of the problem of characterizing perfect semigroups in general was effected in [7]. The presentation of the result requires some terminology. An abelian involution semigroup H, not necessarily having a zero, is *$*$-archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \boldsymbol{N}$ such that $n(x + x^*) = y + z$. A *$*$-archimedean component* of a $*$-semigroup S is a $*$-archimedean $*$-subsemigroup of S which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components (see [9], Section 4.3, for the case of identical involution). An abelian semigroup T is *torsion-free cancellative* if T is cancellative and the group $T - T$ is torsion-free. Every $*$-semigroup S has a greatest torsion-free cancellative identical-involution $*$-homomorphic image, "greatest" in the sense that the corresponding congruence relation in S is smallest; see [7] for a construction. If S is a $*$-semigroup, for $r \in S$ and $\varphi \in \boldsymbol{C}^S$ define a function $E_r\varphi : S \to \boldsymbol{C}$ by $E_r\varphi(s) = \varphi(r+s)$ for $s \in S$. The function φ is *completely positive definite* if $E_r\varphi \in \mathcal{P}(S)$ for all $r \in S$. Denote by $\mathcal{P}_c(S)$ the set of all completely positive definite functions on S. Denote by S^*_+ the set of all nonnegative characters on S. Denote by $\mathcal{A}(S^*_+)$ the smallest σ-field of subsets of S^*_+ rendering the evaluations measurable. Note that $\mathcal{A}(S^*_+) = \{A \cap S^*_+ | A \in \mathcal{A}(S^*)\}$. Denote by $F_+(S^*_+)$ the set of all measures defined on $\mathcal{A}(S^*_+)$ and integrating the evaluations. Note that $F_+(S^*_+)$ can be identified with the set of those $\mu \in F_+(S^*)$ such that the inner measure $\mu_*(S^* \setminus S^*_+)$ vanishes. A function $\varphi : S \to \boldsymbol{R}_+$ is a *Stieltjes moment function* if $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*_+)$, and a Stieltjes moment function φ is *Stieltjes determinate* if there is only one such μ. Denote by $\mathcal{H}_S(S)$

the set of all Stieltjes moment functions on S, and by $\mathcal{H}_{S,D}(S)$ the subset of Stieltjes determinate Stieltjes moment functions. Then $\mathcal{H}_{S,D}(S) \subset \mathcal{H}_S(S) \subset \mathcal{P}_c(S)$. The $*$-semigroup S is *Stieltjes semiperfect* if $\mathcal{H}_S(S) = \mathcal{P}_c(S)$, and *Stieltjes perfect* if $\mathcal{H}_{S,D}(S) = \mathcal{P}_c(S)$. Every perfect semigroup is Stieltjes perfect ([7], Lemma 3.2). Now [7], Corollary 3.1, reads: *A $*$-semigroup S is perfect if and only if for every $*$-archimedean component H of S the greatest torsion-free cancellative identical-involution $*$-homomorphic image of $H \cup \{0\}$ is Stieltjes perfect.*

Suppose G is an abelian group with involution. Then G is $*$-archimedean. Indeed, given $x, y \in G$, we have $x + x^* = y + (x + x^* - y)$. Thus G has only one $*$-archimedean component, viz., itself. By the above result from [7] it follows that G is perfect if and only if the greatest torsion-free identical-involution $*$-homomorphic image H of G is Stieltjes perfect. The group H can be constructed as follows: Let K be the quotient group $G/\{x - x^* | x \in G\}$. Consider K with the identical involution and note that the quotient mapping is a $*$-homomorphism. Now H can be identified with the quotient group K/K_t where K_t is the torsion of K.

Thus, in order to characterize perfect $*$-groups in general, it suffices to characterize Stieltjes perfect groups among torsion-free abelian groups carrying the identical involution.

A torsion-free abelian group G is *countably free* if every countable subgroup of G is a free abelian group. We shall show that a torsion-free abelian group G is perfect if and only if the greatest countably free homomorphic image of G is perfect.

2. The result

In this section, suppose H is a torsion-free abelian group, and consider every group with the identical involution. Let U be the enveloping rational vector space of H, and consider every rational vector space V with the topology defined by the condition that a subset D of V is open in V if and only if $D \cap W$ is open in W, for the canonical topology on a finite-dimensional rational vector space, for every finite-dimensional linear subspace W of V. For every subset A of U denote by $\bar{A}$ the closure of A in U. Let V be the greatest linear subspace of U contained in $\bar{H}$, let $\pi : U \to U/V$ be the quotient mapping, and define $G = \pi(H)$. The *rank* of a subgroup of a rational vector space is the dimension of the linear subspace spanned by that group. If every subgroup of G of finite rank is free then G is countably free ([11], p. 378).

LEMMA 1. $\bar{H} = \pi^{-1}(G)$.

Proof. We have to show $\bar{H} = H + V$, and it suffices to show that $H + V$ is closed in U. Suppose W is a finite-dimensional linear subspace of U. Then $\overline{(H+V) \cap W}$ is a closed subgroup of W, so by [5], Lemma 2, there exist a linear subspace X of W and a free subgroup F of W/X such that $\overline{(H+V) \cap W} = \rho^{-1}(F)$ where $\rho : W \to W/X$ is the quotient mapping. Cosets of X in $\rho^{-1}(F)$ are open in $\rho^{-1}(F)$. Since $\overline{(H+V) \cap W} = \rho^{-1}(F)$, it follows that $(H + V) \cap W$ intersects every such coset. Since

$$X \subset \rho^{-1}(F) = \overline{(H+V) \cap W} \subset \overline{H+V} \subset \overline{H+\bar{H}} = \bar{H}$$

and since V is the greatest linear subspace of U contained in $\bar{H}$ then $X \subset V$, so $X \subset (H+V) \cap W$. Given a coset A of X in $\rho^{-1}(F)$, by the preceding we can choose $a \in A \cap (H+V) \cap W$. Then $A = a + X \subset (H+V) \cap W$. This shows $\rho^{-1}(F) \subset (H+V) \cap W$. Since $\rho^{-1}(F)\overline{(H+V) \cap W}$, this proves that $(H+V) \cap W$ is closed in W. This being so for every finite-dimensional linear subspace W of U, we have shown that $H+V$ is closed in U. This completes the proof.

LEMMA 2. *G is closed in U/V.*

Proof. Suppose W is a finite-dimensional linear subspace of U/V; we have to show that $G \cap W$ is closed in W. Since $\overline{G \cap W}$ is a closed subgroup of W, by [5], Lemma 2, there exist a linear subspace X of W and a free subgroup F of W/X such that $\overline{G \cap W} = \rho^{-1}(F)$ where $\rho : W \to W/X$ is the quotient mapping. Since cosets of X in $\rho^{-1}(F)$ are open in $\rho^{-1}(F)$, it follows that $\overline{G \cap X} = X$. In particular, $X \subset \bar{G}$, so $\pi^{-1}(X) \subset \pi^{-1}(\bar{G}) \subset \overline{\pi^{-1}(G)} = \bar{\bar{H}} = \bar{H}$ by Lemma 1. Since V is the greatest linear subspace of U contained in $\bar{H}$, it follows that $\pi^{-1}(X) = V$, so $X = \{0\}$. Thus F is a free subgroup of W and $\overline{G \cap W} = F$. Since F is discrete it follows that $G \cap W = F$. This shows that $G \cap W$ is closed in W. This being so for every finite-dimensional linear subspace W of U/V, we have shown that G is closed in U/V. This completes the proof.

LEMMA 3. *G is the greatest countably free torsion-free homomorphic image of H.*

Proof. First suppose E is a subgroup of G of finite rank. If W is the linear subspace of U/V spanned by E then, as in the preceding proof, $G \cap W$ is free, so its subgroup E is also free. Thus every subgroup of G of finite rank is free, and as already mentioned, by [11], p. 378 it follows that G is countably free.

Now suppose h is a homomorphism of H onto a countably free torsion-free abelian group K. Let W be the enveloping rational vector space of K. There is a unique linear mapping $\tilde{h} : U \to W$ such that $h = \tilde{h}|H$. Since K is closed in W then $\tilde{h}^{-1}(K)$ is closed in U. Since $H \subset \tilde{h}^{-1}(K)$, it follows that $\bar{H} \subset \tilde{h}^{-1}(K)$ and in particular $V \subset \tilde{h}^{-1}(K)$, that is, $\tilde{h}(V) \subset K$. Since $\tilde{h}(V)$ is a vector space and since K is countably free, it follows that $\tilde{h}(V) = \{0\}$, that is, $V \subset \ker \tilde{h}$, hence $H \cap \ker \pi = H \cap V \subset H \cap \ker \tilde{h} = \ker h$, which shows that K is a homomorphic image of G in a canonical way. This completes the proof.

THEOREM 1. *H is perfect if and only if G is.*

Proof. For a torsion-free abelian group carrying the identical involution, perfectness is equivalent to Stieltjes perfectness, by [7], Corollary 3.1. If H is perfect, so is G, being the homomorphic image of H under π. Now suppose G is perfect; we have to show that H is perfect, or equivalently, Stieltjes perfect. Suppose φ is a completely positive definite function on H; we have to show that φ is a Stieltjes determinate Stieltjes moment function. This can be proved by showing that φ extends to a unique completely positive definite function on $\bar{H}$ and that $\bar{H}$ is Stieltjes perfect. By [5], Lemma 4, every completely positive definite function on a subgroup of a finite-dimensional rational vector space extends to a continuous function on the closure with respect to the canonical topology on a finite-dimensional rational vector space. Let (K, Φ) be a pair consisting of a subgroup K

of $\bar{H}$ containing H and a completely positive definite function Φ on K extending, φ, and maximal for the ordering of such pairs defined by the condition that $(K', \Phi') \leq (K'', \Phi'')$ if and only if $K' \subset K''$ and $\Phi' = \Phi''|K'$. (Existence of (K, Φ) is guaranteed by Zorn's Lemma, but can also be shown without it.) For every finite-dimensional linear subspace X of U, the function $\Phi|(K \cap X)$ extends (by [5], Lemma 4) to a continuous, hence completely positive definite, function Φ_X on $\overline{K \cap X}$. If X and Y are finite-dimensional linear subspaces of U satisfying $X \subset Y$ then $\Phi_X|(K \cap X) = \Phi|(K \cap X) = \Phi_Y|(K \cap X)$, so by continuity, $\Phi_X = \Phi_Y|\overline{K \cap X}$. This fact permits us to define a function $\bar{\Phi}$ on $\bigcup_X \overline{K \cap X}$ (the union indexed by the set of all finite-dimensional linear subspaces of U) by the condition that $\Phi_X = \bar{\Phi}|\overline{K \cap X}$ for every finite-dimensional linear subspace X of U. Since each Φ_X is completely positive definite, and since only finitely many values at a time enter into the definition of complete positive definiteness, then $\bar{\Phi}$ is completely positive definite. Since $\bigcup_X \overline{K \cap X}$ contains K, and since $\bar{\Phi}$ extends Φ, by the maximality of (K, Φ) it follows that $K = \bigcup_X \overline{K \cap X}$. Thus, if X is a finite-dimensional linear subspace of U then $\overline{K \cap X} = K \cap X$, i.e., $K \cap X$ is closed in X. This being so for every finite-dimensional linear subspace X of U, the group K is closed in U. Since $H \subset K$, it follows that $\bar{H} \subset K$, so $K = \bar{H}$. We have shown that φ extends to a completely positive definite function on $\bar{H}$. Uniqueness of the extension follows from the fact that completely positive definite functions are continuous, cf. [5], Lemma 4.

It remains to be shown that $\bar{H}$ is Stieltjes perfect, or equivalently, perfect. But if W is a linear complement of V in U then $\bar{H} \cap W$ is isomorphic to G, so $\bar{H}$ is isomorphic to $V \times G$, which is a product of two perfect groups, hence perfect. This completes the proof.

REMARK 1. We conjecture that a non-perfect $*$-group G is semiperfect if and only if there is a surjective $*$-homomorphism $h : G \to \mathbf{Z}$ ($\mathbf{Z}$ being considered with the identical involution) such that the kernel of h is a perfect $*$-subgroup of G.

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